IMPURITY DISPERSION IN THE TURNER MODEL FOR FLOW IN A POROUS CHANNEL

V. I. Maron

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A one-dimensional model for longitudinal diffusion in a capillary with an immobile layer is derived.

In the Turner model for a porous channel, the useful flow cross section is smaller than the total cross section of the system, since a cylindrical liquid layer at the periphery is assumed immobile. With this model it is possible to study impurity-transport effects in porous media due to the existence of dead-end pores, whose contents participate in the mass transfer.

The axisymmetric concentration distribution in this system is described by a transport equation of the type

$$\frac{\partial c}{\partial t} + u(r)\frac{\partial c}{\partial x} = \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right) + D\frac{\partial^2 c}{\partial x^2}, \quad c = c(t, x, r),\ t>0,\ 0<r<R,\ x>0, \tag{1}$$

where

$$u(r) = \begin{cases} 0, & a<r<R, \\ 2U_0\left(1-\dfrac{r^2}{a^2}\right), & 0<r<a. \end{cases} \tag{2}$$

At the inner surface of the capillary there is no flow of material, so that

$$\left(\frac{\partial c}{\partial r}\right)_{r=R} = 0.$$

We introduce the concentration averaged over the entire cross section

$$\theta_m = \frac{2}{R^2}\int_0^R rc\,dr \tag{3}$$

and we construct a one-dimensional model of the diffusion in a flow with velocity profile (2) for this concentration. We proceed as in [2, 3]. We find

$$\frac{\partial\theta_m}{\partial\tau} + \Upsilon\frac{\partial\theta_m}{\partial\xi} = \left(1 + \Upsilon^2\sum_{n=1}^{\infty}\frac{a_n^2}{\lambda_n^2 J_0^2(\lambda_n)}\right)\frac{\partial^2\theta_m}{\partial\xi^2},$$

$$a_n = 2\int_0^{a^2/4R^2}(1-4z)\,J_0(2\lambda_n\sqrt{z})\,dz, \tag{4}$$

$$z = \frac{r^2}{4R^2},\quad \xi = \frac{x}{R},\quad \tau = \frac{Dt}{R^2},\quad \Upsilon = \frac{U_0 R}{D}.$$

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The effective coefficient K_* in the one-dimensional diffusion model in the Turner system, (4), depends on the ratio of the useful radius to the total radius.

This is one of the aspects which distinguish longitudinal transport in a Turner system from that in a capillary; this point is mentioned in [1].

A more fundamental aspect of the description of impurity transport in the Turner system on the basis of the one-dimensional model is that the quantity θ_m does not give a complete picture of the distribution of the impurity concentration in the system. In this system there are two regions in which the mass transport occurs in different manners.

The existence of a retarding layer leads to mass-transfer conditions different from those in the inner part of the flow. For example, the mixing in the useful cross-section results from two mechanisms: convective transport, with a velocity varying along the radius, and diffusion. In the fixed layer, on the other hand, the mass transfer is due solely to diffusion.

In each of these regions (in the immobile layer and in the flow core) there is a scale time for equalization of the concentration inhomogeneities over a transverse cross section; these times can be quite different from each other.

In the model described by (4), as in the model proposed in [1], there is no parameter which takes mass transfer between the immobile layer and the useful flow cross section into account effectively.

To introduce such a parameter into the one-dimensional model for longitudinal mixing, we consider two average concentrations in the immobile layer and in the useful flow cross section. We define these concentrations, θ_f and θ_s, respectively, by

$$\theta_f = \frac{2}{R^2 - a^2}\int_a^R rcdr,\ \theta_s = \frac{2}{a^2}\int_0^a rcdr. \tag{5}$$

We are primarily interested in determining the latter concentration, since it is this concentration, rather than θ_m, which corresponds to the mass distribution at the exit from the system.

It is not difficult to show that the functions θ_f, θ_s, and θ_m are related linearly:

$$\rho^2\theta_s + (1-\rho^2)\theta_f = \theta_m,\ \rho = a/R. \tag{6}$$

Here θ_m is assumed to be known on the basis of solution of Eq. (4) for certain initial and boundary conditions corresponding to the conditions under which the mass transfer occurs. To find the relationship between the unknown functions θ_f and θ_s, we assume that the distribution of θ_f reproduces that of θ_s, with a certain time lag τ_*. More precisely, we set

$$\theta_f(\tau,\ \xi) = u(\tau - \tau_*)\,\theta_s(\tau - \tau_*,\ \xi). \tag{7}$$

The time lag τ_* is yet another parameter which (along with the effective diffusion coefficient) corresponds to mass transfer; this time lag is introduced in the derivation of the one-dimensional diffusion model for an effective account of mass transfer between the immobile layer and the useful flow cross section. There are two ways to determine the magnitude of this time lag: the first is based on an analysis of the corresponding experimental data with the help of solutions obtained on the basis of the one-dimensional model; the second is based on a more detailed analysis of the mass transfer between the immobile layer and the flow and a solution of the three-dimensional problem describing this mass transfer.

To illustrate the second method of determining τ_* we consider the following problem: we assume that τ_f and τ_s are the scale times for the equalization of the concentration inhomogeneities in the immobile layer and in the useful cross section, respectively. We introduce yet another scale time, τ_0, which is the time over which there is an important change in the concentration θ_s. This scale time is governed by the ratio of the length of the mixture region to the average flow velocity.

We assume the following inequalities among these scale times:

$$\tau_0 \gg \tau_f \gg \tau_s. \tag{8}$$

Under these conditions the relaxation time over which the inhomogeneities in the concentration distribution in the immobile layer are equalized can be found by solving the problem giving an approximation description of the mass transfer between the flow and the layer:

$$\frac{\partial c_f}{\partial \tau} = \frac{\partial^2 c_f}{\partial \eta^2},\ c_f = c_f(\tau,\ \eta),\ \tau > 0,\ 0 < \eta < \Delta = 1 - \rho,$$
$$c_f|_{\tau=0} = 0,\ c_f|_{\eta=\Delta} = \theta_s,\ \frac{\partial c_f}{\partial \eta}\Big|_{\eta=0} = 0. \tag{9}$$

From the solution of this problem we find the relaxation time

$$\tau_* = \lambda_1^{-2},\ \lambda_1 = \frac{\pi}{2\Delta}, \tag{10}$$

where λ_1 is the maximum eigenvalue of the Sturm – Liouville problem which arises in the solution of (9),

$$X_n''(\eta) + \lambda_n^2 X_n(\eta) = 0,\ X_n(0) = X_n'(0) = 0.$$

Substituting θ_f from (7) into (6),

$$\rho^2 \theta_s(\tau,\ \xi) + (1 - \rho^2)\, u(\tau - \tau_*)\, \theta_s(\tau - \tau_*,\ \xi) = \theta_m, \tag{11}$$

we find an equation for the function θ_s.

To solve Eq. (7), we use a Laplace – Carson transformation:

$$\rho^2 \Omega_s + (1 - \rho^2) \exp(-\tau_* p)\, \Omega_s = \Omega_m,$$
$$\Omega_s = \Omega_s(p,\ \xi),\ \Omega_m = \Omega_m(p,\ \xi), \tag{12}$$

where

$$\Omega = p \int_0^\infty \theta(\tau,\ \xi) \exp(-p\tau)\, d\tau.$$

From (12) we find the transform Ω_s:

$$\Omega_s = \frac{1}{\rho^2} \frac{\Omega_m}{1 + \alpha \exp(-\tau_* p)};\ \alpha = \frac{1 - \rho^2}{\rho^2}. \tag{13}$$

We see from this expression that the concentration θ_s differs from θ_m and that it approaches the function θ_m only asymptotically, in the limit $\tau \to \infty$ ($p \to 0$).

We write the second factor on the right side as a series in powers of $\alpha \exp(-\tau_* p)$, $\alpha < 1$:

$$\Omega_s = \frac{1}{\rho^2} \sum_{n=0}^{\infty} (-1)^n \alpha^n \exp(-n\tau_* p)\, \Omega_m. \tag{14}$$

Using the retardation theorem to determine the inverse transform, we find that

$$\theta_s = \frac{1}{\rho^2} \sum_{n=0}^{\infty} (-1)^n \alpha^n \theta_m(\tau - n\tau_*,\ \xi),$$
$$\theta_m(\tau \leqslant 0,\ \xi) \equiv 0,\ \alpha < 1. \tag{15}$$

This analysis should be thought of as an example of constructing a model for longitudinal transport in which there is a parameter other than the diffusion coefficient: the time lag. The time lag effectively incorporates the features of the mass transfer near the inner surface of the tube.

In the Turner model the capillaries have stagnation zones; in flow through a capillary without stagnation zones, the flow velocity near the inner surface is nevertheless much smaller than that near the axis. A similar circumstance was noted by Taylor, who assumed that long "tails" are formed near the inner surface as a result of slow erosion of impurities [4].

The method outlined above for constructing a model with two parameters – the diffusion coefficient and the time lag – can be extended to this case.

NOTATION

c is the local impurity concentration;

θ_m is the concentration averaged over the entire cross section of the system;

θ_s is the concentration averaged over the useful cross section;
θ_f is the average concentration in the immobile layer;
$u(r)$ is the velocity profile;
U_0 is the average velocity;
D is the molecular diffusion coefficient;
Ω is the transform of the function θ;
$u(\tau)$ is the unit step function;
K_* is the effective diffusion coefficient;

$$K_* = 1 + \Upsilon^2 \sum_{n=1}^{\infty} \frac{a_n^2}{\lambda_n^2 J_0^2(\lambda_n)};$$

$\Upsilon = U_0 R / D$ is the dimensionless complex;
$J_\nu(x)$ is the Bessel function of the first kind;
λ_n is the positive root of the equation $J_1(\lambda_n) = 0$;
a is the radius of the useful cross section;
R is the radius of the system;
$\rho = a / R$;
$\Delta = 1 - \rho$;
$\alpha = (1 - \rho^2) / \rho^2$;
$\tau = Dt / R^2$ is the dimensionless time;
τ_*, τ_f, τ_s, τ_0 are the scale times;
x, r, y are the spatial variables;
η is the dimensionless distance y / R from the boundary of the fixed layer.

LITERATURE CITED

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